

# The ordered Beta-distribution

## Parametrisation

Let us first summarize the Beta-distribution. It has the following density

$$p(y) = \frac{1}{B(a, b)} y^{a-1} (1-y)^{b-1}, \quad 0 < y < 1, \quad a > 0, \quad b > 0$$

where  $B(a, b)$  is the Beta-function

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

and  $\Gamma(x)$  is the Gamma-function. The (re-)parameterisation used is

$$\mu = \frac{a}{a+b}, \quad 0 < \mu < 1$$

and

$$\phi = a+b, \quad \phi > 0,$$

as it makes

$$E(y) = \mu \quad \text{and} \quad \text{Var}(y) = \frac{\mu(1-\mu)}{1+\phi}.$$

The parameter  $\phi$  is known as the *precision parameter*, since for fixed  $\mu$ , the larger  $\phi$  the smaller the variance of  $y$ . The parameters  $\{a, b\}$  are given as  $\{\mu, \phi\}$  as follows,

$$a = \mu\phi \quad \text{and} \quad b = -\mu\phi + \phi.$$

The **ordered Beta**-likelihood (reference here<sup>1</sup>) is a modification which includes a derived model for  $y = 0$  and  $y = 1$ . Let  $\mu = g(\eta)$ , where  $\eta$  is the linear predictor (see below) and  $g(\cdot)$  is the inverse link-function, then

$$p_o(y) = \begin{cases} 1 - g(\eta - k_1) & \text{for } y = 0 \\ (g(\eta - k_1) - g(\eta - k_2))p(y) & \text{for } 0 < y < 1 \\ g(\eta - k_2) & \text{for } y = 1 \end{cases}$$

with two offset-parameters  $(k_1, k_2)$  where  $k_1 < k_2$ .

## Link-function

The linear predictor  $\eta$  is linked to the mean  $\mu$  using a default logit-link

$$\mu = \frac{1}{1 + \exp(-\eta)}.$$

## Hyperparameter

The hyperparameter are the precision parameter  $\phi$ , which is represented as

$$\phi = s_i \exp(\theta_1)$$

where  $s = (s_i) > 0$  is a fixed scaling. The offset parameters are expressed as

$$k_1 = \theta_1 - \exp(\theta_2), \quad k_2 = \theta_1 + \exp(\theta_2)$$

The prior is defined on  $(\theta_1, \theta_2, \theta_3)$ .

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<sup>1</sup><https://doi.org/10.1017/pan.2022.20>

## Specification

- `family="obeta"`
- Required argument:  $y$
- Optional argument:  $s$  (argument `scale`, default all 1,  $s > 0$ )

## Hyperparameter spesification and default values

`doc` The ordered Beta likelihood

`hyper`

`theta1`

```
hyperid 61101
name precision parameter
short.name phi
output.name precision-parameter for the obeta observations
output.name.intern intern precision-parameter for the obeta observations
initial 2.30258509299405
fixed FALSE
prior loggamma
param 1 0.1
to.theta function(x) log(x)
from.theta function(x) exp(x)
```

`theta2`

```
hyperid 61102
name offset location
short.name loc
output.name offset location-parameter for the obeta observations
output.name offset location-parameter for the obeta observations
initial 0
fixed FALSE
prior normal
param 0 10
to.theta function(x) x
from.theta function(x) x
```

`theta3`

```
hyperid 61103
name offset width
short.name width
output.name offset width-parameter for the obeta observations
output.name offset width-parameter for the obeta observations
initial 0
fixed FALSE
prior loggamma
param 1 1
```

```

to.theta function(x) log(x)
from.theta function(x) exp(x)

```

```
survival FALSE
```

```
discrete FALSE
```

```
link default logit loga cauchit probit cloglog ccloglog loglog
```

```
pdf obeta
```

## Example

In the following example we estimate the parameters in a simulated example.

```
library(INLA)
```

```

robeta <- function(eta, k1, k2, precision)
{
  stopifnot(k1 < k2)
  g <- function(...) inla.link.invlogit(...)
  p <- cbind(1 - g(eta - k1), g(eta - k1) - g(eta - k2), g(eta - k2))
  eta <- as.vector(eta)
  n <- length(eta)
  group <- numeric(n)
  x <- numeric(n)
  for(i in 1:n) {
    group[i] <- sample(1:3, 1, prob = p[i, ])
    if (group[i] == 1) {
      x[i] <- 0
    } else if (group[i] == 3) {
      x[i] <- 1
    } else {
      mu <- g(eta[i])
      a <- mu * precision
      b <- -mu * precision + precision
      x[i] <- rbeta(1, a, b)
    }
  }
  return (x)
}

```

```

n <- 10^4
x <- rnorm(n, sd = 0.3)
eta <- 1 + x
k1 <- -2
k2 <- 2
prec <- 2
y <- robeta(eta, k1, k2, prec)

```

```

r <- inla(y ~ 1 + x,
  data = data.frame(y, x),
  family = "obeta",

```

```

        control.inla = list(cmin = 0),
        verbose = TRUE)
summary(r)

## produce posteriors for k1 and k2
xx <- inla.hyperpar.sample(10^5, r, intern = FALSE)
loc <- xx[, "offset location-parameter for the obeta observations"]
width <- xx[, "offset width-parameter for the obeta observations"]
k1.mc <- loc - lwidth
k2.mc <- loc + lwidth
par(mfrow = c(1, 2))
hist(k1.mc, prob=TRUE, n = 100); abline(v=k1, lwd = 5, col = "blue")
hist(k2.mc, prob=TRUE, n = 100); abline(v=k2, lwd = 5, col = "blue")

```

## Notes

None.