

# User defined integration points

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## Introduction

This short note describe a new option that allow the user to use user-defined integration points (or “design” points), instead of the default ones. The relevant integration in INLA does

$$\int f(x|\theta, y) \pi(\theta|y) d\theta = f(x|y)$$

where  $\pi(\theta|y)$  is the approximated posterior marginal for the hyperparameters, and where  $f(x|\theta, y)$  is the approximated marginal for  $x$  for that configuration. The output of this integral is the posterior marginal  $f(x|y)$ . In practice, we use a discrete set of integration points for  $\theta$ , and corresponding weights  $w$ , to get

$$f(x|y) \approx \sum_i f(x|\theta_i, y) w_i \pi(\theta_i|y)$$

for which we require  $w_i \geq 0$  and  $\sum_i w_i = 1$ . Usually, the integration is done in a *standardised scale*,

$$z = A(\theta - \gamma)$$

i.e. with respect to  $\pi(z|y)$ . Here,  $\gamma$  is the mode of  $\pi(\theta|y)$  and the matrix  $A$  is the negative square root of the approximated covariance matrix for  $\theta|y$  at the mode.

The relevant options are

```
opts = control.inla(int.strategy = "user", int.design = Design)
```

where **Design** is a matrix with the integration points and the integration weights. The  $j$ th row of **Design** consists of the values  $\theta_j = (\theta_{1j}, \dots, \theta_{mj})$ , and the integration weight for this configuration,  $w_j$ . The values are in the  $\theta$ -scale, meaning that you have to know exactly what you are doing, including knowing the ordering of the hyperparameters.

Another version, is to define the points in the standardised scale  $z$ . To do this, use

```
opts = control.inla(int.strategy = "user.std", int.design = Design)
```

instead. The meaning of **Design** is unchanged, except that these can be given in standardised coordinates. This version is more relevant if you want to implement a generic new integration design instead of the ones already provided.

## First example

In this artificial example, we want to compute the change of the marginal variance of one component,  $x_1$ , of a hidden AR(1) process, with respect to lag one correlation  $\rho$ . So we want to compute

$$\frac{\partial \text{SD}(x_1|y)}{\partial \rho}$$

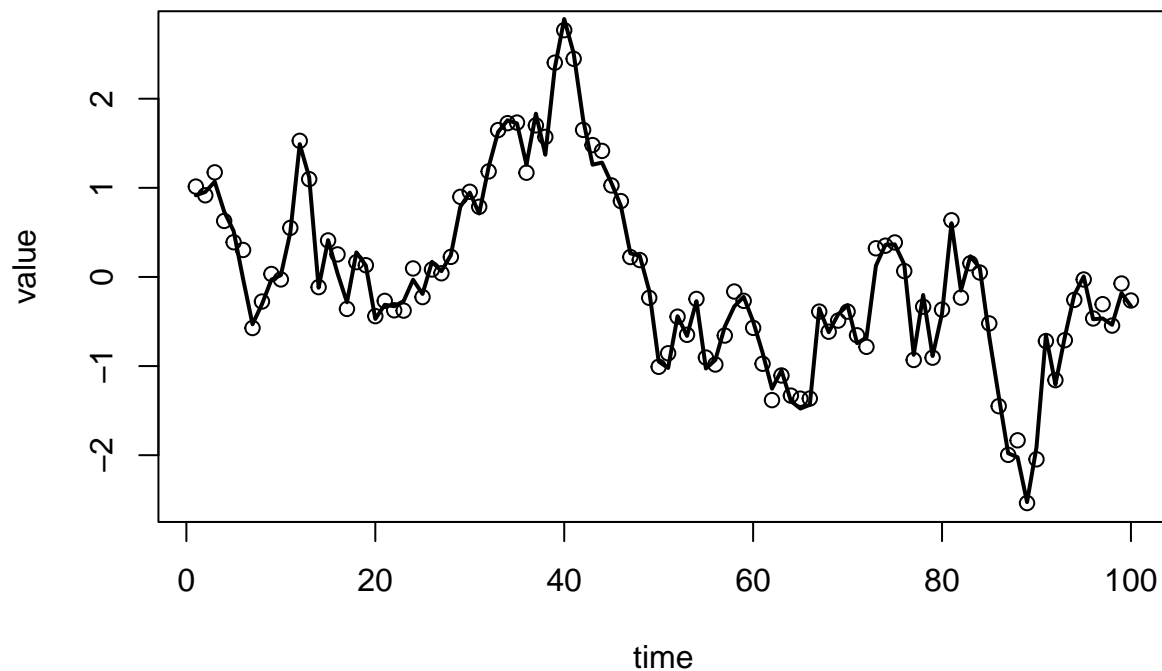
for a fixed value of  $\rho = \rho_0$ . We have to compute a numerical approximation, using finite difference. While doing this, it is a good idea to keep the design fixed, to avoid introducing an error for changing that part as well.

Let us first setup the experiment

```
n = 100
rho = 0.9
x = scale(arima.sim(n, model = list(ar = rho)))
y = x + rnorm(n, sd = 0.1)
```

this gives the following

```
plot(y, xlab = "time", ylab = "value")
lines(x, lwd=2)
```



To compute the derivative, we do

```
rho.0 = rho
to.theta = inla.models()$latent$ar1$hyper$theta2$to.theta
rho.0.internal = to.theta(rho.0)

r = inla(y ~ -1 + f(time, model="ar1",
  hyper = list(
    theta1 = list(prior = "loggamma",
      param = c(1,1)),
    theta2 = list(initial = rho.0.internal,
      fixed=TRUE))),
  control.inla = list(int.strategy = "grid"),
```

```
data = data.frame(y, time = 1:n))
```

```
sd.0 = r$summary.random$time[1,"sd"]
print(sd.0)
```

```
## [1] 0.01360281
```

We will now change  $\rho$  a little, while we keep the same integration points,

```
summary(r)
```

```
##
```

```
## Call:
```

```
## c("inla.core(formula = formula, family = family, contrasts = contrasts,
## ", " data = data, quantiles = quantiles, E = E, offset = offset, ", "
## scale = scale, weights = weights, Ntrials = Ntrials, strata = strata,
## ", " lp.scale = lp.scale, link.covariates = link.covariates, verbose =
## verbose, ", " lincomb = lincomb, selection = selection, control.compute
## = control.compute, ", " control.predictor = control.predictor,
## control.family = control.family, ", " control.inla = control.inla,
## control.fixed = control.fixed, ", " control.mode = control.mode,
## control.expert = control.expert, ", " control.hazard = control.hazard,
## control.lincomb = control.lincomb, ", " control.update =
## control.update, control.lp.scale = control.lp.scale, ", "
## control.pardiso = control.pardiso, only.hyperparam = only.hyperparam,
## ", " inla.call = inla.call, inla.arg = inla.arg, num.threads =
## num.threads, ", " keep = keep, working.directory = working.directory,
## silent = silent, ", " inla.mode = inla.mode, safe = FALSE, debug =
## debug, .parent.frame = .parent.frame)" )
```

```
## Time used:
```

```
## Pre = 0.231, Running = 0.278, Post = 0.00956, Total = 0.519
```

```
## Random effects:
```

```
## Name Model
```

```
## time AR1 model
```

```
##
```

```
## Model hyperparameters:
```

```
## mean sd 0.025quant 0.5quant
```

```
## Precision for the Gaussian observations 1.68e+04 1.61e+04 1074.325 1.15e+04
```

```
## Precision for time 8.46e-01 1.18e-01 0.632 8.39e-01
```

```
## 0.975quant mode
```

```
## Precision for the Gaussian observations 59821.26 2954.322
```

```
## Precision for time 1.10 0.828
```

```
##
```

```
## Marginal log-Likelihood: -70.19
```

```
## is computed
```

```
## Posterior summaries for the linear predictor and the fitted values are computed
```

```
## (Posterior marginals needs also 'control.compute=list(return.marginals.predictor=TRUE)')
```

```
nm <- nrow(r$summary.hyperpar)
```

```
Design = as.matrix(cbind(r$joint.hyper[, seq_len(nm)], 1))
```

```
head(Design)
```

```
## Log precision for the Gaussian observations Log precision for time 1
```

```
## [1,] 9.868498 -0.16719624 1
```

```
## [2,] 8.984204 -0.17041102 1
```

```
## [3,] 10.752792 -0.16398146 1
```

```
## [4,]          9.868879        -0.27200481  1
## [5,]          9.868117        -0.06238767  1
## [6,]          8.984585        -0.27521959  1
```

where the last column is the (un-normalised) integration weights. The integration weights should depend on the area associated with every point, but for simplicity we just set the weights to be a constant.

Design has dimension 70, 3. We call `inla()` again reusing the previous found mode

```
h.rho = 0.01
rho.1.internal = to.theta(rho.0 + h.rho)
rr = inla(y ~ -1 + f(time, model="ar1",
  hyper = list(
    theta1 = list(prior = "loggamma",
      param = c(1,1)),
    theta2 = list(initial = rho.1.internal,
      fixed=TRUE))),
  control.mode = list(result = r, restart=FALSE),
  data = data.frame(y, time = 1:n),
  control.inla = list(
    int.strategy = "user",
    int.design = Design))
sd.1 = rr$summary.random$time[1,"sd"]
print(sd.1)
```

```
## [1] 0.01263927
```

and then our estimate of the derivative is

```
deriv.1 = (sd.1 - sd.0) / h.rho
print(deriv.1)
```

```
## [1] -0.09635385
```

PS: In the logfile of the `inla()`-call, the configurations are shown in the  $z$  scale even for `int.strategy="user"`.

## Second example

There is also another (experimental) option, that is

```
control.inla = list(int.strategy = "user.expert")
```

for which the weights **includes**  $\pi(\theta_i|y)$ . The following example show how to use it, fitting the same model in three different ways.

```
n = 50
x = rnorm(n)
y = 1 + x + rnorm(n, sd = 0.2)
param = c(1, 0.04)
dz = 0.1
r.std = inla(y ~ 1 + x, data = data.frame(y, x),
  control.inla = list(int.strategy = "grid",
    dz = dz,
    diff.logdens = 8),
  control.family = list(
    hyper = list(
      prec = list(
        prior = "loggamma",
        param = param))))
```

```

s = r.std$internal.summary.hyperpar[1,"sd"]
m = r.std$internal.summary.hyperpar[1,"mean"]
theta = m + s*seq(-4, 4, by = dz)
weight = dnorm(theta, mean = m, sd = s)

r = rep(list(list()), length(theta))
for(k in seq_along(r)) {
  r[[k]] = inla(y ~ 1 + x,
               control.family = list(
                 hyper = list(
                   prec = list(
                     initial = theta[k],
                     fixed=TRUE))),
               data = data.frame(y, x))
}
r.merge = inla.merge(r, prob = weight)

## Warning in inla.merge(r, prob = weight): This function is experimental.
## Warning in inla.merge(r, prob = weight): Merging 'cpo' and 'pit'-results
## are/can be, approximate only

r.design = inla(y ~ 1 + x,
               data = data.frame(y, x),
               control.family = list(
                 hyper = list(
                   prec = list(
                     ## the prior here does not really matter, as we will override
                     ## it with the user.expert in any case.
                     prior = "pc.prec",
                     param = c(1, 0.01))),
               control.inla = list(int.strategy = "user.expert",
                                   int.design = cbind(theta, weight)))

for(k in 1:2) {
  plot(inla.s marginal(r.std$marginals.fixed[[k]]),
       lwd = 2, lty = 1, type = "l",
       xlim = inla.qmarginal(c(0.0001, 0.9999), r.std$marginals.fixed[[k]]))
  lines(inla.s marginal(r.design$marginals.fixed[[k]]),
        lwd = 2, col = "blue", lty = 1)
  lines(inla.s marginal(r.merge$marginals.fixed[[k]]),
        lwd = 2, col = "yellow", lty = 1)
}

```

