Zero-inflated models: Poisson, Binomial, negative Binomial and BetaBinomial

Parametrisation

There is support two types of zero-inflated models, which we name type 0 and type 1. These are defined for both the Binomial, the Poisson, the censored Poisson, the negative Binomial and BetaBinomial likelihood. For simplicity we will describe only the Poisson as the other cases are similar.

Type 0

The (type 0) likelihood is defined as

$$\text{Prob}(y \mid \ldots) = p \times 1_{[y=0]} + (1-p) \times \text{Poisson}(y \mid y > 0)$$

where p is a hyperparameter where

$$p = \frac{\exp(\theta)}{1+\exp(\theta)}$$

and θ is the internal representation of p; meaning that the initial value and prior is given for θ. This is model is called `zeroinflatedpoisson0` (and `zeroinflatedbinomial0`).

Type 1

The (type 1) likelihood is defined as

$$\text{Prob}(y \mid \ldots) = p \times 1_{[y=0]} + (1-p) \times \text{Poisson}(y)$$

where p is a hyperparameter where

$$p = \frac{\exp(\theta)}{1+\exp(\theta)}$$

and θ is the internal representation of p; meaning that the initial value and prior is given for θ. This is model is called `zeroinflatedpoisson1` (and `zeroinflatedbinomial1`).

Link-function

As for the Poisson, the Binomial the negative Binomial and the BetaBinomial.

Hyperparameters

For Poisson and the Binomial, there is one hyperparameter; where

$$p = \frac{\exp(\theta)}{1+\exp(\theta)}$$

and the prior and initial value is is given for θ.

For the negative Binomial and BetaBinomial, there are two hyperparameters. The overdispersion parameter n for the negative Binomial is represented as

$$\theta_1 = \log(n)$$

and the prior is defined on θ_1. The zero-inflation parameter p, is represented as

$$p = \frac{\exp(\theta_2)}{1+\exp(\theta_2)}$$

and the prior and initial value is is given for θ_2. For the BetaBinomial it is similar.

Specification

- family = `zeroinflatedbinomial0`
- family = `zeroinflatedbinomial1`
- family = `zeroinflatednbinomial0`
- family = `zeroinflatednbinomial1`
- family = `zeroinflatedpoisson0`
- family = `zeroinflatedpoisson1`
- family = `zeroinflatedcenpoisson0`
- family = `zeroinflatedcenpoisson1`
- family = `zeroinflatedbetabinomial0`
- family = `zeroinflatedbetabinomial1`
- Required arguments: As for the Binomial, the negative Binomial, BetaBinomial and Poisson likelihood.

Hyperparameter spesification and default values

Zeroinflated Binomial Type 0

doc Zero-inflated Binomial, type 0

hyper

theta

hyperid 90001
name logit probability
short.name prob
initial -1
fixed FALSE
prior gaussian
param -1 0.2
to.theta `function(x) log(x / (1 - x))`
from.theta `function(x) exp(x) / (1 + exp(x))`

survival FALSE

discrete FALSE

link default logit loga cauchit probit cloglog loglog robit sn

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Zeroinflated Binomial Type 1

doc Zero-inflated Binomial, type 1

hyper

theta

hyperid 91001
name logit probability
short.name prob
initial -1
fixed FALSE
prior gaussian
param -1 0.2
to.theta `function(x) log(x / (1 - x))`
from.theta `function(x) exp(x) / (1 + exp(x))`

survival FALSE

discrete FALSE

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Zeroinflated NegBinomial Type 0

doc Zero inflated negBinomial, type 0

hyper

theta1

hyperid 95001
name log size
short.name size
initial 2.30258509299405
fixed FALSE
prior pc.mgamma
param 7
to.theta `function(x) log(x)`
from.theta `function(x) exp(x)`

theta2

hyperid 95002
name logit probability
short.name prob
initial -1
fixed FALSE
prior gaussian
param -1 0.2
to.theta `function(x) log(x / (1 - x))`

from.theta `function(x) exp(x) / (1 + exp(x))`

survival FALSE

discrete FALSE

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Zeroinflated NegBinomial Type 1

doc Zero inflated negBinomial, type 1

hyper

theta1

hyperid 96001
name log size
short.name size
initial 2.30258509299405
fixed FALSE
prior pc.mgamma
param 7
to.theta `function(x) log(x)`
from.theta `function(x) exp(x)`

theta2

hyperid 96002
name logit probability
short.name prob
initial -1
fixed FALSE
prior gaussian
param -1 0.2
to.theta `function(x) log(x / (1 - x))`
from.theta `function(x) exp(x) / (1 + exp(x))`

survival FALSE

discrete FALSE

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Zeroinflated BetaBinomial Type 0

doc Zero-inflated Beta-Binomial, type 0

hyper

theta1

hyperid 88001
name overdispersion
short.name rho
initial 0
fixed FALSE
prior gaussian
param 0 0.4
to.theta `function(x) log(x / (1 - x))`
from.theta `function(x) exp(x) / (1 + exp(x))`

theta2

hyperid 88002
name logit probability
short.name prob
initial -1
fixed FALSE
prior gaussian
param -1 0.2
to.theta `function(x) log(x / (1 - x))`
from.theta `function(x) exp(x) / (1 + exp(x))`

survival FALSE

discrete TRUE

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Zeroinflated BetaBinomial Type 1

doc Zero-inflated Beta-Binomial, type 1

hyper

theta1

hyperid 89001
name overdispersion
short.name rho
initial 0
fixed FALSE
prior gaussian
param 0 0.4
to.theta `function(x) log(x / (1 - x))`
from.theta `function(x) exp(x) / (1 + exp(x))`

theta2

hyperid 89002
name logit probability
short.name prob

initial -1
fixed FALSE
prior gaussian
param -1 0.2
to.theta `function(x) log(x / (1 - x))`
from.theta `function(x) exp(x) / (1 + exp(x))`

survival FALSE

discrete TRUE

link default logit loga cauchit probit cloglog loglog robit sn

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Zeroinflated Poisson Type 0

doc Zero-inflated Poisson, type 0

hyper

theta

hyperid 85001
name logit probability
short.name prob
initial -1
fixed FALSE
prior gaussian
param -1 0.2
to.theta `function(x) log(x / (1 - x))`
from.theta `function(x) exp(x) / (1 + exp(x))`

survival FALSE

discrete FALSE

link default log

pdf zeroinflated

Zeroinflated Poisson Type 1

doc Zero-inflated Poisson, type 1

hyper

theta

hyperid 86001
name logit probability
short.name prob
initial -1
fixed FALSE
prior gaussian

param -1 0.2
to.theta `function(x) log(x / (1 - x))`
from.theta `function(x) exp(x) / (1 + exp(x))`

survival FALSE

discrete FALSE

link default log

pdf zeroinflated

Zeroinflated Censored Poisson Type 0

doc Zero-inflated censored Poisson, type 0

hyper

theta

hyperid 87101
name logit probability
short.name prob
initial -1
fixed FALSE
prior gaussian
param -1 0.2
to.theta `function(x) log(x / (1 - x))`
from.theta `function(x) exp(x) / (1 + exp(x))`

status experimental

survival FALSE

discrete FALSE

link default log

pdf zeroinflated

Zeroinflated Censored Poisson Type 1

doc Zero-inflated censored Poisson, type 1

hyper

theta

hyperid 87201
name logit probability
short.name prob
initial -1
fixed FALSE
prior gaussian
param -1 0.2

to.theta `function(x) log(x / (1 - x))`

from.theta `function(x) exp(x) / (1 + exp(x))`

status experimental

survival FALSE

discrete FALSE

link default log

pdf zeroinflated

Example

In the following example we estimate the parameters in a simulated example for both type 0 and type 1.

Poisson

```
## type 0
n=100
a = 1
b = 1
z = rnorm(n)
eta = a + b*z
p = 0.2
E = sample(c(1,5,10,15), size=n, replace=TRUE)
lambda = E*exp(eta)

## first sample y|y>0
y = rpois(n, lambda = lambda)
is.zero = (y == 0)
while(sum(is.zero) > 0)
{
    y[is.zero] = rpois(sum(is.zero), lambda[is.zero])
    is.zero = (y == 0)
}
## then set some of these to zero
y[ rbinom(n, size=1, prob=p) == 1 ] = 0

data = list(y=y,z=z)
formula = y ~ 1+z
result0 = inla(formula, family = "zeroinflatedpoisson0", data = data, E=E)
summary(result0)

## type 1
y = rpois(n, lambda = lambda)
y[ rbinom(n, size=1, prob=p) == 1 ] = 0
data = list(y=y,z=z)
formula = y ~ 1+z
result1 = inla(formula, family = "zeroinflatedpoisson1", data = data, E=E)
summary(result1)
```

Binomial

```
## type 0
n=100
a = 1
b = 1
z = rnorm(n)
eta = a + b*z
p = 0.2
Ntrials = sample(c(1,5,10,15), size=n, replace=TRUE)
prob = exp(eta)/(1 + exp(eta))

y = rbinom(n, size = Ntrials, prob = prob)
is.zero = (y == 0)
while(sum(is.zero) > 0)
{
    y[is.zero] = rbinom(sum(is.zero), size = Ntrials[is.zero], prob = prob[is.zero])
    is.zero = (y == 0)
}
y[ rbinom(n, size=1, prob=p) == 1 ] = 0
data = list(y=y,z=z)
formula = y ~ 1+z
result0 = inla(formula, family = "zeroinflatedbinomial0", data = data, Ntrials = Ntrials)
summary(result0)

## type 1
y = rbinom(n, size = Ntrials, prob = prob)
y[ rbinom(n, size=1, prob=p) == 1 ] = 0
data = list(y=y,z=z)
formula = y ~ 1+z
result1 = inla(formula, family = "zeroinflatedbinomial1", data = data, Ntrials=Ntrials)
summary(result1)
```

Advanced example

In the following example we estimate the parameters in a simulated example for a type0 likelihood, where one linear predictor enters the zero-probability and one other linear predictor enters the non-zero Poisson for example. The same trick can be used for other models of type0. The trick is that the likelihood

$$p^* 1_{[y=0]} + (1 - p^*) P(y|y > 0)$$

can be reformulated as a Bernoulli likelihood for the "class"-variable

$$z = \begin{cases} 1, & \text{if } \ y = 0 \\ 0, & \text{if } \ y > 0. \end{cases}$$

where p^* is the probability for success, and zero-inflated type0 likelihood (with fixed $p = 0$) for those $y > 0$. Since p^* and the linear predictor in P is separated into two likelihoods, we can apply one linear predictor to each one, hence extend the basic model to cases where p^* also depends on a linear predictor. Here is a small simulated example doing this.

```
require(INLA)

n = 100
a = 0.5
b = 1.5
x1 = rnorm(n, sd = 0.5)

eta.z = -a - b*x1
z = rbinom(n, 1, inla.link.logit(eta.z, inverse=TRUE))
n.y = sum(z)

x2 = rnorm(n.y, sd = 0.5)
eta.y = a + b*x2
lambda = exp(eta.y)
y = rpois(n.y, lambda)

is.zero = (y == 0)
while(sum(is.zero) > 0)
{
   y[is.zero] = rpois(sum(is.zero), lambda[is.zero])
   is.zero = (y == 0)
}

Y = matrix(NA, n + n.y, 2)
Y[1:n, 1] = z
Y[n + 1:n.y, 2] = y

form = Y ~ 0 + mu.z + mu.y + cov.z + cov.y
ldat = list(
        Y=Y,
        mu.z=rep(1:0, c(n, n.y)),
        mu.y=rep(0:1, c(n, n.y)),
        cov.z=c(x1, rep(NA,n.y)),
        cov.y=c(rep(NA, n), x2))
```

```
res <- inla(form, data=ldat,
            family=c('binomial', 'zeroinflatedpoisson0'),
            control.family=list(
                    list(),
                    list(hyper = list(
                                prob = list(
                                        initial = -20,
                                        fixed = TRUE)))))
round(res$summary.fix, 4)
```

Notes

None.

Extentions

There are some extentions available which currently is only implemented for the cases where its needed/requested.

Type 2 Is like Type 1 but where (for the Poisson)

$$p = 1 - \left(\frac{E\exp(x)}{1 + E\exp(x)}\right)^{\alpha}$$

where $\alpha > 0$ is the hyperparameter instead of p (and $E\exp(x)$ is the mean). Available for Poisson as `zeroinflatedpoisson2`, for binomial as `zeroinflatedbinomial2` and for the negative binomial as `zeroinflatednbinomial2`.

The internal representation is $\theta = \log(\alpha)$ and prior is defined on $\log(\alpha)$.

Zeroinflated Poisson Type 2

doc Zero-inflated Poisson, type 2

hyper

theta

hyperid 87001
name log alpha
short.name a
initial 0.693147180559945
fixed FALSE
prior gaussian
param 0.693147180559945 1
to.theta `function(x) log(x)`
from.theta `function(x) exp(x)`

survival FALSE

discrete FALSE

link default log

pdf zeroinflated

Zeroinflated Binomial Type 2

doc Zero-inflated Binomial, type 2

hyper

theta

hyperid 92001
name alpha
short.name alpha
initial -1
fixed FALSE
prior gaussian
param -1 0.2
to.theta `function(x) log(x)`
from.theta `function(x) exp(x)`

survival FALSE

discrete FALSE

link default logit loga cauchit probit cloglog loglog robit sn

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Zeroinflated Negative Binomial Type 2

doc Zero inflated negBinomial, type 2

hyper

theta1

hyperid 99001
name log size
short.name size
initial 2.30258509299405
fixed FALSE
prior pc.mgamma
param 7
to.theta `function(x) log(x)`
from.theta `function(x) exp(x)`

theta2

hyperid 99002
name log alpha
short.name a
initial 0.693147180559945
fixed FALSE
prior gaussian
param 2 1
to.theta `function(x) log(x)`

from.theta `function(x) exp(x)`

survival FALSE

discrete FALSE

link default log

pdf zeroinflated

Zeroinflated Negative Binomial Type 1 Strata 2

doc Zero inflated negBinomial, type 1, strata 2

hyper

theta1

hyperid 97001
name log size
short.name size
initial 2.30258509299405
fixed FALSE
prior pc.mgamma
param 7
to.theta `function(x) log(x)`
from.theta `function(x) exp(x)`

theta2

hyperid 97002
name logit probability 1
short.name prob1
initial -1
fixed FALSE
prior gaussian
param -1 0.2
to.theta `function(x) log(x / (1 - x))`
from.theta `function(x) exp(x) / (1 + exp(x))`

theta3

hyperid 97003
name logit probability 2
short.name prob2
initial -1
fixed FALSE
prior gaussian
param -1 0.2
to.theta `function(x) log(x / (1 - x))`
from.theta `function(x) exp(x) / (1 + exp(x))`

theta4

hyperid 97004

name logit probability 3

short.name prob3

initial -1

fixed TRUE

prior gaussian

param -1 0.2

to.theta `function(x) log(x / (1 - x))`

from.theta `function(x) exp(x) / (1 + exp(x))`

theta5

hyperid 97005

name logit probability 4

short.name prob4

initial -1

fixed TRUE

prior gaussian

param -1 0.2

to.theta `function(x) log(x / (1 - x))`

from.theta `function(x) exp(x) / (1 + exp(x))`

theta6

hyperid 97006

name logit probability 5

short.name prob5

initial -1

fixed TRUE

prior gaussian

param -1 0.2

to.theta `function(x) log(x / (1 - x))`

from.theta `function(x) exp(x) / (1 + exp(x))`

theta7

hyperid 97007

name logit probability 6

short.name prob6

initial -1

fixed TRUE

prior gaussian

param -1 0.2

to.theta `function(x) log(x / (1 - x))`

from.theta `function(x) exp(x) / (1 + exp(x))`

theta8

hyperid 97008

name logit probability 7

short.name prob7

initial -1

fixed TRUE

prior gaussian

param -1 0.2

to.theta `function(x) log(x / (1 - x))`

from.theta `function(x) exp(x) / (1 + exp(x))`

theta9

hyperid 97009

name logit probability 8

short.name prob8

initial -1

fixed TRUE

prior gaussian

param -1 0.2

to.theta `function(x) log(x / (1 - x))`

from.theta `function(x) exp(x) / (1 + exp(x))`

theta10

hyperid 97010

name logit probability 9

short.name prob9

initial -1

fixed TRUE

prior gaussian

param -1 0.2

to.theta `function(x) log(x / (1 - x))`

from.theta `function(x) exp(x) / (1 + exp(x))`

theta11

hyperid 97011

name logit probability 10

short.name prob10

initial -1

fixed TRUE

prior gaussian

param -1 0.2

to.theta `function(x) log(x / (1 - x))`

from.theta `function(x) exp(x) / (1 + exp(x))`

status experimental

survival FALSE

discrete FALSE

link default log

pdf zeroinflated

Zeroinflated Negative Binomial Type 1 Strata 3

doc Zero inflated negBinomial, type 1, strata 3

hyper

- **theta1**
  - **hyperid** 98001
  - **name** logit probability
  - **short.name** prob
  - **initial** -1
  - **fixed** FALSE
  - **prior** gaussian
  - **param** -1 0.2
  - **to.theta** `function(x) log(x / (1 - x))`
  - **from.theta** `function(x) exp(x) / (1 + exp(x))`
- **theta2**
  - **hyperid** 98002
  - **name** log size 1
  - **short.name** size1
  - **initial** 2.30258509299405
  - **fixed** FALSE
  - **prior** pc.mgamma
  - **param** 7
  - **to.theta** `function(x) log(x)`
  - **from.theta** `function(x) exp(x)`
- **theta3**
  - **hyperid** 98003
  - **name** log size 2
  - **short.name** size2
  - **initial** 2.30258509299405
  - **fixed** FALSE
  - **prior** pc.mgamma
  - **param** 7
  - **to.theta** `function(x) log(x)`
  - **from.theta** `function(x) exp(x)`
- **theta4**
  - **hyperid** 98004
  - **name** log size 3
  - **short.name** size3
  - **initial** 2.30258509299405
  - **fixed** TRUE
  - **prior** pc.mgamma
  - **param** 7
  - **to.theta** `function(x) log(x)`
  - **from.theta** `function(x) exp(x)`

theta5

hyperid 98005
name log size 4
short.name size4
initial 2.30258509299405
fixed TRUE
prior pc.mgamma
param 7
to.theta `function(x) log(x)`
from.theta `function(x) exp(x)`

theta6

hyperid 98006
name log size 5
short.name size5
initial 2.30258509299405
fixed TRUE
prior pc.mgamma
param 7
to.theta `function(x) log(x)`
from.theta `function(x) exp(x)`

theta7

hyperid 98007
name log size 6
short.name size6
initial 2.30258509299405
fixed TRUE
prior pc.mgamma
param 7
to.theta `function(x) log(x)`
from.theta `function(x) exp(x)`

theta8

hyperid 98008
name log size 7
short.name size7
initial 2.30258509299405
fixed TRUE
prior pc.mgamma
param 7
to.theta `function(x) log(x)`
from.theta `function(x) exp(x)`

theta9

hyperid 98009
name log size 8
short.name size8

initial 2.30258509299405
fixed TRUE
prior pc.mgamma
param 7
to.theta `function(x) log(x)`
from.theta `function(x) exp(x)`

theta10

hyperid 98010
name log size 9
short.name size9
initial 2.30258509299405
fixed TRUE
prior pc.mgamma
param 7
to.theta `function(x) log(x)`
from.theta `function(x) exp(x)`

theta11

hyperid 98011
name log size 10
short.name size10
initial 2.30258509299405
fixed TRUE
prior pc.mgamma
param 7
to.theta `function(x) log(x)`
from.theta `function(x) exp(x)`

status experimental

survival FALSE

discrete FALSE

link default log

pdf zeroinflated

0.0.1 Zero and N-inflated Binomial likelihood: type 3

This is the case where

$$\begin{aligned}\text{Prob}(y|\ldots) &= p_0 \times 1_{[y=0]} + \\ &\quad p_N \times 1_{[y=N]} + \\ &\quad (1 - p_0 - p_N) \times \text{binomial}(y, N, p)\end{aligned}$$

where:

$$p = \frac{\exp(\eta)}{1+\exp(\eta)} \qquad p_0 = \frac{p^{\alpha_0}}{1 + p^{\alpha_0} + (1-p)^{\alpha_N}} \qquad p_N = \frac{(1-p)^{\alpha_N}}{1 + p^{\alpha_0} + (1-p)^{\alpha_N}}$$

There are 2 hyperparameters, α_0 and α_N, governing zero-inflation where: The zero-inflation parameters α_0 and α_N are represented as $\theta_0 = \log(\alpha_0)$; $\theta_N = \log(\alpha_N)$ and the prior and initial value is given for θ_0 and θ_N respectively.

Here is an example

```
nsim<-10000
x<-rnorm(nsim)
alpha0<-1.5
alphaN<-2.0
p = exp(x)/(1+exp(x))
p0 = p^alpha0 / (1 + p^alpha0 + (1-p)^alphaN)
pN = (1-p)^alphaN / (1 + p^alpha0 + (1-p)^alphaN)
P<-cbind(p0, pN, (1-p0 -pN))
N<-rpois(nsim,20)
y<-rep(0,nsim)
for(i in 1:nsim)
    y[i]<-sum(rmultinom(1,size = 1,P[i,])*c(0,N[i],rbinom(1,N[i],p[i])))
formula = y ~1 + x
r = inla(formula, family = "zeroninflatedbinomial3",  Ntrials = N, verbose = TRUE,
           data = data.frame(y, x))
```

and the default settings

doc Zero and N inflated binomial, type 3

hyper

- **theta1**
  - **hyperid** 93101
  - **name** alpha0
  - **short.name** alpha0
  - **initial** 1
  - **fixed** FALSE
  - **prior** loggamma
  - **param** 1 1
  - **to.theta** `function(x) log(x)`
  - **from.theta** `function(x) exp(x)`
- **theta2**
  - **hyperid** 93102
  - **name** alphaN
  - **short.name** alphaN
  - **initial** 1
  - **fixed** FALSE
  - **prior** loggamma
  - **param** 1 1
  - **to.theta** `function(x) log(x)`
  - **from.theta** `function(x) exp(x)`

status experimental

survival FALSE

discrete FALSE

link default logit loga cauchit probit cloglog loglog robit sn

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