

# The z-model

## Parametrization

The z-model is an implementation of the “classical” way to define the “random effect” part of a mixed model, through

$$\eta = \dots + Zz$$

where  $Z$  is a  $n \times m$  matrix and  $z$  a vector of length  $m$  representing zero-mean “random effects”. The z-model is defined as the augmented model

$$\tilde{z} = \begin{pmatrix} v \\ z \end{pmatrix}$$

where  $v \sim \mathcal{N}_n(Zz, \kappa I)$ , where  $\kappa$  is a high fixed precision, and where the precision matrix for  $z$  is  $\tau C$  where  $C > 0$  is a  $m \times m$  (fixed) matrix and  $\tau$  is the precision parameter.

## Hyperparameters

The precision parameter of the z-model is represented as

$$\theta = \log(\tau)$$

and prior is assigned to  $\theta$ . The parameter  $\kappa$  is kept fixed at all times.

## Specification

The z-model is specified inside the `f()` function as

```
f(<whatever>, model="z", Z = <Z>, Cmatrix = <Cmat>, hyper = <hyper>,  
  precision = <precision>)
```

where the required `Z`-matrix argument defines the  $Z$  matrix. The (optional) `Cmatrix` defines the  $C$  matrix and is by default taken to be the diagonal matrix with dimension  $m$ . The `precision` parameter defines the value of  $\kappa$ , and `hyper` the hyperparameter specification for  $\tau$ .

If  $Z$  is a  $n \times m$  matrix then the  $C$  matrix must be  $m \times m$  matrix, and  $\tilde{z}$  has length  $n + m$ . The  $n$  first terms of  $\tilde{z}$  is  $v$  and the last  $m$  terms of  $\tilde{z}$  is  $z$ .

If `constr=TRUE` is given, then this is defined as  $\sum_{i=1}^m z_i = 0$ . If `extraconstr` is given, then it is applied to  $\tilde{z}$ , hence `extraconstr$A` must be a  $k \times (n + m)$  matrix where  $k$  is the number of linear constraints.

## Hyperparameter specification and default values

`doc` The z-model in a classical mixed model formulation

`hyper`

```
theta  
  hyperid 31001  
  name log precision  
  short.name prec  
  initial 4  
  fixed FALSE  
  prior loggamma  
  param 1 5e-05
```

```

    to.theta function(x) log(x)
    from.theta function(x) exp(x)

constr FALSE

nrow.ncol FALSE

augmented FALSE

aug.factor 1

aug.constr

n.div.by

n.required TRUE

set.default.values TRUE

pdf z

```

## Example 1

```

## Example for implementing the standard mixed model
##   y = X*%beta + Z*%b+e
## with b ~N(0,tau * Q) With Q a precision matrix

library(INLA)
set.seed(123)
n = 100
nz = 5
nb = 4
tmp = matrix(rnorm(nz^2), nz, nz)
Qz = tmp %*% t(tmp)
Z = matrix(runif(n*nz), n, nz)
b = inla.qsample(1, Q=Qz)
beta = 5 + rnorm(nb)
X = matrix(runif(n*nb), n, nb)
X[,1] = 1 # want an intercept

eta = X %*% beta + Z %*% b
y = eta + rnorm(n, sd=0.01)
Qx = diag(nb)

formula = y ~ -1 + X + f(id.z, model="z", Cmatrix=Qz, Z=Z)
result = inla(formula, data = list(y=y, id.z = 1:n, X=X))

plot(b, result$summary.random$id.z$mean[-(1:n)], pch=19)
abline(0, 1)

```

## Example 2

```

## An example demonstrating two ways to implement the model eta = Z*z,
## where z ~ N(0, tau*Q).

## Simulate data
n = 100
m = 10

```

```

Z = matrix(rnorm(n*m), n, m)
rho = 0.8
Qz = toeplitz(rho^(0:(m-1)))
prec.fixed = FALSE ## the precision parameter for z
z = inla.qsample(1, Q=Qz)
eta = Z %*% z
s = 0.1 ## noise stdev
s.fixed = TRUE
y = eta + rnorm(n, sd = s)

## This is normally not needed at all, but it demonstrate how to set
## the 'high precisions': in the z-model, in the A-part of the linear
## predictor, and in the linear predictor iteself.
precision = exp(15)

## The first approach use the z-model.
r = inla(y ~ -1 + f(idx, model="z", Z=Z,
                    precision=precision,
                    Cmatrix=Qz,
                    hyper = list(
                        prec = list(
                            initial = 0,
                            fixed = prec.fixed,
                            param = c(1, 1)))))
data = list(y=y, idx=1:n),
control.family = list(
    hyper = list(
        prec = list(
            initial = log(1/s^2),
            fixed=s.fixed))),
control.predictor = list(
    compute=TRUE,
    precision=precision,
    initial = log(precision)))

## The second one uses the A-matrix
rr = inla(y ~ -1 + f(idx, model="generic",
                    precision = precision,
                    Cmatrix=Qz,
                    hyper = list(
                        prec = list(
                            initial = 0,
                            fixed = prec.fixed,
                            param = c(1, 1)))))
data = list(y=y, idx=1:m),
control.family = list(
    hyper = list(
        prec = list(
            initial = log(1/s^2),
            fixed=s.fixed))),
control.predictor = list(
    compute=TRUE,
    A=Z,
    precision=precision,
    initial = log(precision)))

## Plot some results

```

```

par(mfrow=c(2, 2))
plot(r$summary.linear.predictor$mean[1:n], eta,
     main="z-model: (eta.estimated, eta)")
plot(r$summary.linear.predictor$mean[1:n], eta,
     main="generic-model: (eta.estimated, eta)")
plot(r$internal.marginals.hyperpar[[1]],
     main="Prec.param (both)")
lines(rr$internal.marginals.hyperpar[[1]])

## compare (log) marginal likelihood. recall to add the missing part,
## see inla.doc("generic")
print(r$mlik - (rr$mlik + 0.5*log(det(Qz))))

r = inla.hyperpar(r)
rr = inla.hyperpar(rr)
plot(r$internal.marginals.hyperpar[[1]],
     main="Prec.param (improved, both)")
lines(rr$internal.marginals.hyperpar[[1]])

## compare (log) marginal likelihood. recall to add the missing part,
## see inla.doc("generic")
print(r$mlik - (rr$mlik + 0.5*log(det(Qz))))

```

### Example 3

```

## This example demonstrate how to use the z-model with intrinsic
## models. The z-model must be proper, which we have to mimic if we
## are using an intrinsic model

## Simulate some data
n = 100
idx = 1:n
x = sin(idx / n * 4 * pi)
s = 0.1
y = x + rnorm(n, sd=s)

## Parameters for the loggamma prior
prior = c(1, 0.001)

## A small constant we add to the diagonal to prevent the model to be
## intrinsic.
d = 1e-8

## RW1
r = inla(y ~ -1 + f(idx, model="rw1", param=prior,
                    constr=TRUE, diagonal=d),
        data = data.frame(y, idx),
        control.family = list(
            hyper = list(
                prec = list(
                    initial = log(1/s^2),
                    fixed = TRUE))))

C = toeplitz(c(2, -1, rep(0, n-2)))
C[1, 1] = C[n, n] = 1
## We must add the extra diagonal contribution here, as otherwise it

```

```

## applies to the hole model (v, z)
diag(C) = diag(C) + d
Z = diag(n)
rr = inla(y ~ -1 + f(idx, model="z", Z=Z,
                  Cmatrix = C, constr=TRUE, param=prior),
          data = data.frame(y, idx),
          control.family = list(
            hyper = list(
              prec = list(
                initial = log(1/s^2),
                fixed = TRUE))))

##
par(mfrow=c(2, 2))
plot(idx, r$summary.random$idx$mean)
lines(idx, rr$summary.random$idx$mean[1:n])
title("rw1: idx")
plot(r$internal.marginals.hyperpar[[1]])
lines(rr$internal.marginals.hyperpar[[1]])
title("rw1: log.prec")

## RW2
r = inla(y ~ -1 + f(idx, model="rw2",
                  ## we cannot define the rankdef for the z-model, but it will
                  ## be set to 1 as constr=TRUE. so we're using the same here,
                  ## even though the rankdef is 0, since we added 'd' on the
                  ## diagonal.
                  rankdef = 1,
                  param=prior, constr=TRUE, diagonal = d),
          data = data.frame(y, idx),
          control.family = list(
            hyper = list(
              prec = list(
                initial = log(1/s^2),
                fixed = TRUE))))

C = toeplitz(c(2, -1, rep(0, n-3), -1))
C = C[-c(1, n), ]
C = t(C) %*% C
## We must add the extra diagonal contribution here, as otherwise it
## applies to the hole model (v, z)
diag(C) = diag(C) + d
Z = diag(n)
rr = inla(y ~ -1 + f(idx, model="z", Z=Z, Cmatrix = C,
                  constr=TRUE, param=prior),
          data = data.frame(y, idx),
          control.family = list(
            hyper = list(
              prec = list(
                initial = log(1/s^2),
                fixed = TRUE))))

##
plot(idx, r$summary.random$idx$mean)
lines(idx, rr$summary.random$idx$mean[1:n])
title("rw2: idx")
plot(r$internal.marginals.hyperpar[[1]])
lines(rr$internal.marginals.hyperpar[[1]])
title("rw2: log.prec")

```

## Notes

None.