The BYM-case

$$y = \ldots + \underbrace{u}_{\text{space}} + \underbrace{v}_{\text{iid.}}$$

Prec $(u) = \tau\, Q$ when gen.var $(u) = 1$

Prec $(v) = \tau\, \cancel{I}\, R$

Rewrite this as

$$y = \ldots + \frac{1}{\sqrt{\tau}}\left(\sqrt{1-\phi}\, v + \sqrt{\phi}\, u\right)$$

So that $\phi = 0$ corresponds to an iid model

Since $\text{Var}\left(\sqrt{1-\phi}\, v + \sqrt{\phi}\, u\right) = 1-\phi + \phi = 1$

then the prior for τ remains unchanged.

So the reference is the distribution for v, ie. $N(0, I)$.

and as ϕ increase, then we mix in dependency all the way to $\phi = 1$. Note that the actual way of parametrisation of ϕ, as $\sqrt{1-\phi}$ and $\sqrt{\phi}$ does not matter. {as long as $\text{Var}(\sqrt{\cdot}\, v + \sqrt{\cdot}\, u) = 1$}.

$$\sqrt{1-\phi}\, v + \sqrt{\phi}\, u \sim N\left(0,\ \text{Var} = (1-\phi) I + \phi\, \underbrace{Q^{-}}_{\text{gen.inv proper scaled}}\right)$$

So we need the KL between

$$\text{Var} = (1-\phi) I + \phi\, Q^{-1}$$

and

$$\text{Var} = I$$

dense matrix, which makes this a bit awkward for large dim(Q).

we need for the KL, to compute

$$\left| (1-\varrho) I + \varrho Q^{-} \right|$$

or, of course,

$$\left| \left[(1-\varrho) I + \varrho Q^{-} \right]^{-1} \right|$$

Now using that

$$(I + A^{-1})^{-1} = A(A+I)^{-1} \qquad [149, \text{ in } MC]$$

we get

$$\left((1-\varrho) I + \varrho Q^{-1} \right)^{-1} = \left[(1-\varrho) \left\{ I + \frac{\varrho}{1-\varrho} Q^{-} \right\} \right]^{-1}$$

$$= \left[(1-\varrho) \left\{ I + \left(\frac{1-\varrho}{\varrho} Q \right)^{-1} \right\} \right]^{-1}$$

$$= \frac{1}{1-\varrho} \Big(I + \underbrace{\left(\frac{1-\varrho}{\varrho} Q \right)}_{A}{}^{-1} \Big)^{-1}$$

$$= \frac{1}{1-\varrho} \left[\frac{1-\varrho}{\varrho} Q \left(\frac{1-\varrho}{\varrho} Q + I \right)^{-1} \right]$$

$$= \frac{1}{\varrho} Q \left(\frac{1-\varrho}{\varrho} Q + I \right)^{-1}$$

So

$$\left| ( )^{-1} \right| = \frac{1}{\varrho^r} |Q| \Big/ \left| \frac{1-\varrho}{\varrho} Q + I \right|$$

So

$$\boxed{\left| (1-\varrho) I + \varrho Q^{-1} \right| = \frac{\left| \frac{1-\varrho}{\varrho} Q + I \right|}{\frac{1}{\varrho^r} |Q|}}$$

Some details.

$$\frac{1}{\sqrt{\tau}}\left(\sqrt{1-\rho}\; v + \sqrt{\rho}\; u\right)$$

when $u \sim N(0, Q_2)$ $[Q_2 \text{ scaled}]$ an $v \sim \mathcal{N}(0, I)$.

So let $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ where $w_2 = u$,

$w_1 | w_2 \sim N\left(\sqrt{\frac{\rho}{\tau}}\, w_2,\ \frac{\tau}{1-\rho} I\right)$, then $w_1 \overset{d}{=} \frac{1}{\sqrt{\tau}}\left(\sqrt{1-\rho}\, v + \sqrt{\rho}\, u\right)$

and $\pi(w)$ is found from.

$$\cancel{\exp}\ -\frac{1}{2} w_2^T Q_2 w_2 - \frac{1}{2}\left(w_1 - \sqrt{\frac{\rho}{\tau}}\, w_2\right)^T \left[\frac{\tau}{1-\rho} I\right]\left(w_1 - \sqrt{\frac{\rho}{\tau}}\, w_2\right)$$

$$= -\frac{1}{2} w_2^T \left[Q_2 + \frac{\rho}{\tau}\frac{\tau}{1-\rho} I\right] w_2 - \frac{1}{2} w_1^T \left[\frac{\tau}{1-\rho} I\right] w_1$$

$$-\frac{1}{2}\left[-2\sqrt{\frac{\rho}{\tau}}\frac{\tau}{1-\rho}\, w_2^T w_1\right]$$

$$= -\frac{1}{2}\left[w_1^T\ w_2^T\right]^T \begin{bmatrix} \frac{\tau}{1-\rho} I & -\frac{\sqrt{\rho\tau}}{1-\rho} I \\ & Q_2 + \frac{\rho}{1-\rho} I \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

not included.

Norm const $\left(\frac{1}{\sqrt{2\pi}}\right)^{2\cdot n} \cdot \left(\frac{\tau}{1-\rho}\right)^{n/2} \cdot |Q|^{1/2}$